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Manifolds of G_2 holonomy from the $N = 4$ sigma model

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Abstract
Using the two-dimensional (2D) $N = 4$ sigma model, with $U(1)^r$ gauge symmetry, and introducing the ADE Cartan matrices as gauge matrix charges, we build 'toric' hyper-Kahler eight real-dimensional manifolds X_8. Dividing by one toric geometry circle action of X_8 manifolds, we present examples describing quotients $X_7 = \frac{X_8}{U(1)}$ of G_2 holonomy. In particular, for the A_r Cartan matrix, the quotient space is a cone on an S^2 bundle over r intersecting $\mathbf{WCP}^2_{(1,2,1)}$ weighted projective spaces according to the A_r Dynkin diagram.

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1. Introduction

Over the past few years, there has been an increasing interest in studying string dualities. One of the important consequences of these studies is that all superstring models are equivalent in the sense that they correspond to different limits in moduli space of the same theory, called M-theory [1–3]. The latter, which is considered nowadays as the best candidate for the unification of the weak and strong coupling sectors of superstring models, is described, at low energies, by an eleven-dimensional supergravity theory.

More recently, a special interest has been shown in the compactification of the M-theory on seven real manifolds X_7 with non-trivial holonomy, in order that these manifolds provide a potential point of contact with low energy semi-realistic physics from M-theory. In particular, one can obtain four-dimensional (4D) theory with $N = 1$ supersymmetry by compactifying M-theory on $R^{1,3} \times X_7$ where X_7 denotes a seven manifold with G_2 holonomy [4–11][1]. In this regard, 4D $N = 1$ resulting physics models depend on geometric properties of X_7. For instance, if X_7 is smooth, the low energy theory contains, in addition to $N = 1$ supergravity, only Abelian gauge group and neutral chiral multiplets. However, non-Abelian gauge symmetries with chiral fermions can be obtained by considering limits where X_7 develops singularities

[1] The latter is the maximal subgroup of $SO(7)$ which can break the eight-dimensional spinorial representation of $SO(7)$ to the seven fundamental representations of G_2 plus one singlet ($8 \to 7 + 1$).

[10, 11]. For this reason, it is interesting to study M-theory on seven singular manifolds with G_2 holonomy. Following [11], an interesting analysis for building such spaces is to consider the quotient of a conical hyper-Kahler eight manifolds X_8 by a $U(1)$ symmetry. This approach, which is called the unfolding of the singularity, guarantees the G_2 holonomy group of the quotient space $\frac{X_8}{U(1)}$. A remarkable feature of this method, which may be related to two-dimensional (2D) $N = 4$ sigma model Calabi–Yau fourfold construction CY^4, is that the $\frac{X_8}{U(1)}$ space solutions differ by what kind of $U(1)$ symmetry is chosen and moreover the matter fields, in four dimensions, are obtained using techniques similar to those in geometric engineering of quantum field theory [12–15].

The aim of this work is to contribute in this direction by considering models with the $N = 4$ 2D sigma model ADE Cartan matrix gauge charges for building X_8 manifolds. This study is motivated by the following points: (1) Actually, these vector charges go beyond those given in the first example studied in [11], where the matrix charge of the hypermultiplets ϕ_i, under $U(1)^r$ gauge symmetry, was

$$q_i^a = -\delta_{i-1}^a + \delta_i^a \qquad a = 1, \ldots, r. \tag{1.1}$$

(2) They may give an analogue connection appearing between the toric Calabi–Yau geometries, used in string theory compactifications, and the structures of ADE Lie algebras. The latter may lead to an analysis similar to that in the geometric engineering method of quantum field theory embedded in string theory.

It was suggested in [11] that the unfolding of the singularity may be adapted to other examples of X_8 manifolds, in particular toric hyper-Kahler manifolds. In this paper, we would present a new class of these X_8 manifolds, which will be called toric hyper-Kahler eight manifolds X_8, with the Calabi–Yau condition in the sigma model construction given by

$$\sum_i q_i^a = 0. \tag{1.2}$$

We will refer to such manifolds as Calabi–Yau fourfolds. Then we give their quotients by a $U(1)$ group symmetry using toric geometry circle actions. Our way involves two steps: first, we introduce the ADE Cartan matrices as 2D $N = 4$ $U(1)^r$ linear sigma model matrix gauge charges. Second, mimicking the analysis of [11] and using toric geometry circle actions, we discuss the construction of a new class of the quotients $\frac{X_8}{U(1)}$ of G_2 holonomy group. In particular, for the A_r Cartan matrix, the quotient space is a cone on an S^2 bundle over r intersecting $\mathbf{WCP}^2_{(1,2,1)}$ weighted projective spaces according to the A_r Dynkin geometry.

The organization of this paper is as follows: in section 2, we give an overview on aspects of the 2D $N = 4$ linear sigma model. Then we give examples illustrating the field theoretical construction of hyper-Kahler manifolds. In section 3, we introduce the ADE Cartan matrices as matrix gauge charges in the 2D $N = 4$ field theory construction of X_8 manifolds. For the A_r Lie algebra, the moduli space of the classical theory is given by the cotangent bundle over r intersecting $\mathbf{WPC}^2_{(1,2,1)}$ weighted projective spaces according to the A_r Dynkin graph, extending the A_r singularity of K3 surfaces described by $N = 2$ type IIA superstring sigma model used in the geometric engineering method. In section 4, we identify the $U(1)$ symmetry group with the toric geometry circle actions of X_8 to present quotients $X_7 = \frac{X_8}{U(1)}$ of G_2 holonomy. For the A_r Cartan matrix gauge charge, the geometry is a cone on an S^2 bundle over r intersecting $\mathbf{WCP}^2_{(1,2,1)}$ weighted projective spaces according to the A_r Dynkin diagram. Our conclusion will be given in section 5.

2. N = 4 sigma model approach

In this section, we review the main lines of the $N = 4$ sigma model approach for building the hyper-Kahler manifolds involved in the study of superstring, M-theory and F-theory, compactifications, Yang–Mills small instant singularities and more general in supersymmetric field theories with eight supercharges [16–18]. For this purpose, consider 2D $N = 4$ supersymmetric $U(1)^r$ gauge theory with n hypermultiplets ϕ_i $(i = 1, \ldots, n)$ of a matrix charge q_i^a $(a = 1, \ldots, r)$, under $U(1)^r$ gauge symmetry, and r 3-vectors FI coupling $\vec{\xi}_a$. The equations defining the hyper-Kahler moduli space of this classical gauge theory are given by the following D-terms:

$$\sum_i q_a^i\left[\phi_i^\alpha \bar{\phi}_{i\beta} + \phi_{i\beta}\bar{\phi}_i^\alpha\right] = \vec{\xi}_a \vec{\sigma}^\alpha_\beta. \tag{2.1}$$

The double indices (i, α) of ϕ_i^α refer to component field doublets of the n hypermultiplets, and $\vec{\sigma}^\alpha_\beta$ are the traceless 2×2 Pauli matrices. For later use, it is interesting to note the following points:

(1) Equations (2.1) have a formal analogy with the D-flatness equations of 2D $N = 2$ $U(1)^r$ toric sigma model involved in the study of type II superstring compactifications on ALE spaces with ADE singularities [13]. The latter are given by

$$\sum_{i=1}^n q_i^a |x_i|^2 = R_a \qquad a = 1, \ldots, r \tag{2.2}$$

where r is the rank of the ADE Lie algebras and q_i^a, up some details, the minus of the corresponding Cartan matrices satisfying the Calabi–Yau condition

$$\sum_{i=1}^n q_i^a = 0. \tag{2.3}$$

Equation (2.2) has a nice geometrical interpretation in terms of toric geometry. This has been an interesting interplay between 2D $N = 2$ sigma models and toric geometry. In this way, (2.2) has a toric diagram which consists of n vertices $\{v_i\}$, in the standard lattice $\mathbf{Z}^{\mathbf{n-r}}$, satisfying the following constraint equations:

$$\sum_{i=1}^n q_i^a v_i = 0 \qquad a = 1, \ldots, r. \tag{2.4}$$

For instance, if one takes $r = 1$, this space describes the $(n - 1)$-dimensional weighted projective with weights q_i.

(2) For each $U(1)$ factor, there are three real constraint equations transforming as an isotriplet of $SU(2)$ R-symmetry ($SU(2)_R$) acting on the hyper-Kahler structures.

(3) Using the $SU(2)_R$ transformations

$$\phi^\alpha = \varepsilon^{\alpha\beta}\phi_\beta \qquad \varepsilon_{12} = \varepsilon^{21} = 1 \qquad \overline{(\phi^\alpha)} = \overline{\phi}_\alpha \tag{2.5}$$

and replacing the Pauli matrices by their expressions, the identities (2.1) can be split as follows:

$$\sum_{i=1}^n q_i^a \left(\left|\phi_i^1\right|^2 - \left|\phi_i^2\right|^2\right) = \xi_a^3 \tag{2.6}$$

$$\sum_{i=1}^n q_i^a \phi_i^1 \overline{\phi}_i^2 = \xi_a^1 + \mathrm{i}\xi^2{}_a \tag{2.7}$$

$$\sum_{i=1}^{n} q_i^a \phi_i^2 \overline{\phi}_i^1 = \xi_a^1 - \mathrm{i}\xi^2{}_a. \tag{2.8}$$

Note that these equations have features similar to those described in [16] leaving only half the supersymmetry of the gauge model.

(4) After dividing the moduli space of zero energy states of the classical gauge theory (2.1) by the action of the $U(1)^r$ gauge symmetry, we find precisely a toric hyper-Kahler variety $X_{4(k-r)}$ of $4(k-r)$ real dimensions. This construction is called the hyper-Kahler quotient extending the Kahler one involved in 2D $N = 2$ toric sigma model [16–18].

(5) The solutions of equations (2.1) depend on the values of the FI couplings. For the case where $\xi^1 = \xi^2 = 0$ and $\xi^3 > 0$, it is not difficult to see that equations (2.1) describe the cotangent bundle over a toric variety defined by

$$\sum_{i=1}^{n} q_i^a \left|\phi_i^1\right|^2 = \xi_a^3. \tag{2.9}$$

Indeed, if we set all $\phi_i^2 = 0$, the ϕ_i^1, modulo the complexified $U(1)^r$ gauge group, determine a toric variety $\frac{C^n}{C^{*r}}$ of $2(n-r)$ real dimensions, see equations (2.2). The equations (2.6)–(2.7) mean that the ϕ_i^2 define the cotangent fibre directions over the toric variety given by (2.9). To see this feature, we assume that $\xi_a^1 = \xi_a^2 = 0$, $a = 1$ and we set $q_i^a = q_i = 1$, so we have

$$\sum_{i=1}^{n} \left(\left|\phi_i^1\right|^2 - \left|\phi_i^2\right|^2\right) = \xi^3 \tag{2.10}$$

and

$$\sum_{i=1}^{n} \phi_i^1 \overline{\phi}_i^2 = 0. \tag{2.11}$$

Equation (2.10), for $\phi_i^2 = 0$, defines the $\mathbf{CP^{n-1}}$ projective space while equation (2.11) means that ϕ_i^2 parameterizes the cotangent directions on it. In what follows, we give two extra examples illustrating this analysis and reconsidering the example given in [11]. In the first example, we consider a 2D $N = 4$ $U(1)$ linear sigma model with two hypermultiplets of a vector charge $(1, -1)$. The D-flatness conditions of this model read

$$\left(\left|\phi_1^1\right|^2 - \left|\phi_2^1\right|^2\right) - \left(\left|\phi_1^2\right|^2 - \left|\phi_2^2\right|^2\right) = \xi^3 \tag{2.12}$$

$$\phi_1^1 \overline{\phi}_1^2 - \phi_2^1 \overline{\phi}_2^2 = 0 \tag{2.13}$$

$$\phi_1^2 \overline{\phi}_1^1 - \phi_2^2 \overline{\phi}_2^1 = 0. \tag{2.14}$$

Permuting the role of ϕ_2^1 and $\bar{\phi}_2^2$, and making the following field change $\varphi_1 = \phi_1^1$, $\varphi_2 = -\bar{\phi}_2^2$, $\psi_1 = \phi_1^2$ and $\psi_2 = \bar{\phi}_2^1$, the constraint equations (2.12)–(2.14) become

$$(|\varphi_1|^2 + |\varphi_2|^2) - (|\psi_1|^2 + |\psi_2|^2) = \xi^3 \tag{2.15}$$

$$\varphi_1 \overline{\psi}_1 + \varphi_2 \overline{\psi}_2 = 0 \tag{2.16}$$

$$\overline{\varphi_1} \psi_1 + \overline{\varphi_2} \psi_2 = 0 \tag{2.17}$$

and describe a cotangent bundle over a $\mathbf{CP^1}$ projective. In this way, the $\mathbf{CP^1}$ is defined by the following equation:

$$|\varphi_1|^2 + |\varphi_2|^2 = \xi^3. \tag{2.18}$$

Recall in passing that the cotangent bundle over $\mathbf{CP^1}$, which is known by the resolved A_1 singularity of K3 surfaces, is isomorphic to $\frac{C^2}{Z_2}$ and plays a crucial role in the study of the non-perturbative limit of type II superstring dynamics in six and four dimensions [13–15]. The second example we want to consider deals with the generalization of the first one. This concerns a 2D $N = 4$ $U(1)^r$ linear sigma model with $(r + 1)$ hypermultiplets of a matrix charge satisfying (1.1). Using the same procedure, the D-flatness conditions (2.1) become

$$(|\varphi_{a-1}|^2 + |\varphi_a|^2) - (|\psi_{a-1}|^2 + |\psi_a|^2) = \xi_a^3 \tag{2.19}$$

$$\psi_{a-1}\overline{\varphi}_{a-1} + \varphi_a\overline{\psi}_a = 0 \tag{2.20}$$

$$\varphi_{a-1}\overline{\psi}_{a-1} + \psi_a\overline{\varphi}_a = 0. \tag{2.21}$$

The solution of these equations describes the cotangent bundle over r intersecting complex curves $\mathbf{CP^1}$. In the limit when all ξ_a^3 go to zero, the CP^1 shrink and one ends with the A_r singularity of local K3 surfaces. Note that this example has been used in [11] to construct seven real-dimensional manifolds X_7 of G_2 holonomy group from the quotient of X_8 hyper-Kahler eight real-dimensional manifolds by a $U(1)$ group symmetry. These eight-dimensional spaces are obtained using the 2D $N = 4$ $U(1)^{(r-1)}$ linear sigma model with $(r + 1)$ hypermultiplets, where the missing $U(1)$ invariance is explored to get the quotient $\frac{X_8}{U(1)}$ of G_2 holonomy group [11]. In what follows we want to give a new class of X_8 manifolds, which will be called toric hyper-Kahler Calabi–Yau fourfolds ($CY^4 = X_8$) by introducing the ADE Cartan matrices instead of the gauge matrix charge given in equation (1.1).

3. Toric hyper-Kahler eight manifolds with Calabi–Yau condition

We start this section by recalling that complex Calabi–Yau manifolds are the best ingredients for obtaining semi-realistic models of superstrings/M/F-theory [18–20], with minimal supercharges in lower dimensions. In particular for later use, Calabi–Yau fourfolds, compact, non-compact, singular or non-singular, are considered as ways for getting $N = 1$ supersymmetric models in four dimensions from the F-theory compactifications [20, 21]. In M-theory context, compatifications on manifolds of G_2 holonomy can be effectively described by 4D $N = 1$ supersymmetry. Furthermore, from the supersymmetry breaking viewpoint, the above geometries, which preserve both the same supercharges, in particular $\frac{1}{8}$ of initial ones of the uncompactified theory, have a similar role in superstrings and M-theory compactifications. From this physical argument and the string duality results, connecting type IIA and type IIB strings, we think that there are at least two natural questions. The latter are as follows: (1) Does there exist a 4D duality connecting M-theory on manifolds of G_2 holonomy and F-theory on Calabi–Yau fourfolds? (2) Or does there exist a link between the corresponding geometries (manifolds of G_2 holonomy and Calabi–Yau fourfolds)? These questions, which are quite similar to the link between M-theory on manifolds of G_2 holonomy and heterotic strings on Calabi–Yau threefolds, need detailed study. However, here we try to give a modest comment on the second one; while the first one will be dealt with in future work. This comment is based on the following known points:

(i) Manifolds with G_2 holonomy can be constructed as $U(1)$ quotients of eight manifolds.
(ii) The maximal group of automorphisms in eight dimensions is $SO(8)$. Using Dynkin geometries this group, including the $SU(4)$ group, can give the G_2 group.
(iii) Eight manifolds can have hyper-Kahler constructions in terms of the $N = 4$ sigma model.

Combining these points with the Calabi–Yau condition $\sum_i q_i^a = 0$ in the sigma model approach, one may say that seven real-dimensional manifolds of the G_2 holonomy group may be constructed from hyper-Kahler eight manifolds with the Calabi–Yau condition. In what follows, we refer to such manifolds as Calabi–Yau fourfold geometries. In this way, the G_2 manifolds can be obtained using quotients by one finite circle, preserving the supercharges. In the present study, following the ideas of [11], we would discuss the construction of seven-dimensional manifolds with G_2 holonomy group from Calabi–Yau fourfold geometry physics data, but with a different realization of the $U(1)$ group symmetry for obtaining the quotient. This study involves two steps. First we will introduce, in the field theoretical construction of Calabi–Yau fourfolds X_8, the ADE Cartan matrices as 2D $N = 4$ linear sigma model matrix gauge charges. Second, mimicking the method of [11] and using toric geometry circle actions, we will discuss quotients $\frac{X_8}{U(1)}$ of G_2 holonomy group which will be given in the next section. Roughly speaking, the toric hyper-Calabi–Yau fourfolds $CY^4 = X_8$ may be viewed as the moduli space of the 2D $N = 4$ supersymmetric $U(1)^r$ gauge theory with $(r + 2)$ ϕ_i^α hypermultiplets $(4(r + 2 - r) = 8)$ with a matrix charge q_a^i with the Calabi–Yau condition (1.2). In what follows, we will consider a matrix charge going beyond equation (1.1). Our choice will be given by ADE Cartan matrices. For simplicity, we first consider the A_r Lie algebra where the Cartan matrix is given by

$$q_i^a = -2\delta_i^a + \delta_{i-1}^a + \delta_{i+1}^a \qquad a = 1, \ldots, r \tag{3.1}$$

satisfying naturally the Calabi–Yau condition $\sum_i q_a^i = 0$. Putting these equations into the D-flatness equations (2.1), one gets the following system of $3r$ equations:

$$\left(\left|\phi_{a-1}^1\right|^2 + \left|\phi_{a+1}^1\right|^2 - 2\left|\phi_a^1\right|^2\right) - \left(\left|\phi_{a-1}^2\right|^2 + \left|\phi_{a+1}^2\right|^2 - 2\left|\phi_a^2\right|^2\right) = \xi_a \tag{3.2}$$

$$\phi_{a-1}^1\overline{\phi^2}_{a-1} + \phi_{a+1}^1\overline{\phi^2}_{a+1} - 2\phi_a^1\overline{\phi^2}_a = 0 \tag{3.3}$$

$$\phi_{a-1}^2\overline{\phi^1}_{a-1} + \phi_{a+1}^2\overline{\phi^1}_{a+1} - 2\phi_a^2\overline{\phi^1}_a = 0. \tag{3.4}$$

We first solve these equations for the simple example of $U(1)$ gauge theory. Then we will give the result for the $U(1)^r$ gauge model. For $r = 1$, the above equations reduce to

$$\left(\left|\phi_0^1\right|^2 + \left|\phi_2^1\right|^2 - 2\left|\phi_1^1\right|^2\right) - \left(\left|\phi_0^2\right|^2 + \left|\phi_2^2\right|^2 - 2\left|\phi_1^2\right|^2\right) = \xi \tag{3.5}$$

$$\phi_0^1\overline{\phi^2}_0 + \phi_2^1\overline{\phi^2}_2 - 2\phi_1^1\overline{\phi^2}_1 = 0 \tag{3.6}$$

$$\phi_0^2\overline{\phi^1}_0 + \phi_2^2\overline{\phi^1}_2 - 2\phi_1^2\overline{\phi^1}_1 = 0. \tag{3.7}$$

To handle these D-terms equations, it should be interesting to note that they are quite similar to equations (2.10)–(2.11), and also to (2.12)–(2.14). After permuting the role of ϕ_2^1 and $\overline{\phi_2^2}$, equations may be rewritten as

$$\left(\left|\phi_0^1\right|^2 + \left|\phi_2^1\right|^2 + 2\left|\overline{-\phi_1^2}\right|^2\right) - \left(\left|\phi_0^2\right|^2 + \left|\phi_1^2\right|^2 + 2\left|\phi_1^1\right|^2\right) = \xi \tag{3.8}$$

$$\phi_0^1\overline{\phi^2}_0 + \phi_2^1\overline{\phi^2}_2 + 2\phi_1^1\overline{\left(-\phi^2{}_1\right)} = 0 \tag{3.9}$$

$$\phi_0^2\overline{\phi^1}_0 + \phi_2^2\overline{\phi^1}_2 + 2\left(-\phi_1^2\right)\overline{\phi^1}_1 = 0. \tag{3.10}$$

Making the following field changes,

$$\phi_0^1 = \varphi_1 \qquad \phi_0^2 = \psi_1 \qquad \phi_1^1 = \varphi_2 \qquad \phi_1^2 = \psi_2 \qquad -\overline{\phi}_1^2 = \varphi_3 \qquad \overline{\phi}_1^1 = \psi_1$$

the above equations become

$$\left(|\varphi_1|^2 + |\varphi_3|^2 + 2|\varphi_2|^2\right) - \left(|\psi_1|^2 + |\psi_3|^2 + 2|\psi_2|^2\right) = \xi^3 \tag{3.11}$$

$$\varphi_1\overline{\psi_1} + \varphi_3\overline{\psi_3} + 2\varphi_2\overline{\psi_2} = 0 \tag{3.12}$$

$$\overline{\varphi_1}\psi_1 + \overline{\varphi_3}\psi_3 + 2\overline{\varphi_2}\psi_2 = 0. \tag{3.13}$$

Using similar analysis to in the previous section, one notes that the above equations describe a cotangent bundle over $\mathbf{WCP}^2_{(1,2,1)}$ weighted projective space. A way to see this feature is to use the link between the $N = 2$ sigma model and toric geometry technics. Indeed, taking $\psi_1 = \psi_2 = \psi_3 = 0$, equations (3.11)–(3.13) reduce to

$$|\varphi_1|^2 + |\varphi_3|^2 + 2|\varphi_2|^2 = \xi^3 \tag{3.14}$$

which can be encoded in a toric diagram. In this diagram, one has three vectors v_1, v_2 and v_3 in Z^2 lattice such that

$$v_1 + v_3 + 2v_2 = 0 \tag{3.15}$$

where the coefficients of v_i are exactly those of φ_i in (3.14). Note that equation (3.15) describes a particular geometry of that given in (2.4). Using the toric geometry language, equation (3.15) defines naturally a $\mathbf{WCP}^2_{(1,2,1)}$ weighted projective space, where ξ^3 is a Kahler real parameter controlling its seize. Equations (3.11)–(3.13), for generic value of ψ_i, can be interpreted to mean that ψ_i parameterizes the fibre cotangent directions over $\mathbf{WCP}^2_{(1,2,1)}$. Since the subset of (3.11) with $\psi_i = 0$ is a $\mathbf{WCP}^2_{(1,2,1)}$ weighted projective space and $\varphi_1\overline{\psi_1} + \varphi_3\overline{\psi_3} + 2\varphi_2\overline{\psi_2} = 0$ is the analogue of equation (2.11), thus the space of solutions of (3.11)–(3.13) is isomorphic to the cotangent space over $\mathbf{WCP}^2_{(1,2,1)}$, $T^*(\mathbf{WCP}^2_{(1,2,1)})$. In the general case corresponding to the $U(1)^r$ gauge theory, if we take all the ξ_a as non-zero, it is not too difficult to see that equations (3.2)–(3.4) describe the cotangent bundle over r intersecting $\mathbf{WCP}^2_{(1,2,1)}$ weighted projective spaces. This means that the base geometry of the cotangent bundle consists of r intersecting $\mathbf{WCP}^2_{(1,2,1)}$ according to the A_r Dynkin diagram, instead of one projective space in the case of $U(1)$ gauge theory. In the limit that some ξ_a are zero, we obtain a singular geometry. Actually, this geometry may be used to extend the intersecting $\mathbf{CP}^1$ projective spaces of ALE spaces involved in the geometric engineering method of the quantum field theory [13–15]. We will conclude this section by noting that this analysis of the A_r Lie algebra may be extended to the others DE Lie algebras. However, these algebras contain trivalent vertex Dynkin geometries, which complicates the computation. Recall that the trivalent Dynkin geometry involves a central node intersecting three other nodes once; moreover, this geometry has been used in the geometric engineering of quantum field theories, in particular in the introduction of fundamental matters in a chain of SU product gauge group with $N = 2$ bifundamental matters. In toric sigma model approach, the corresponding vector charge, on the Calabi–Yau condition, is given by

$$q_i = (0, \ldots, -2, 1, 1, 1, 0, \ldots, 0, -1)$$

instead of the bivalent geometry (3.1). *A priori* there are different ways one may follow to overcome this problem. A naive way is to delete these trivalent vertices. In this case, the D-flatness constraint equations have similar solutions to that of the A_n Lie algebra. However a tricky method is to leave this and use the trivalent geometry results involved in the elliptic fibrations singularities over the complex plane. In this way, the base geometry of X_8 may be given by three chains of intersecting $\mathbf{WCP}^2_{(1,2,1)}$ according to the trivalent geometry.

In what follows, we will discuss the corresponding seven real-dimensional manifolds of G_2 holonomy group using $U(1)$ quotients. Following the ideas of [11], we should look for a $U(1)$ group symmetry acting on X_8. As mentioned before, there are many ways one may follow to choose the $U(1)$ group action of X_8. In this regard, the solutions differ by what kind of $U(1)$ symmetry is chosen. Two kinds of solutions are given in [11]. But here we will consider another way. The latter is inspired by the toric geometry circle actions.

4. On the quotient space $X_7 = \frac{X_8}{U(1)}$ of G_2 holonomy

Having constructed toric hyper-Kahler Calabi–Yau fourfolds X_8 associated with ADE Cartan matrices sigma model gauge charges, we are now in the position to carry out quotient spaces $X_7 = \frac{X_8}{U(1)}$ of G_2 holonomy group using circle actions involved in toric varieties. Before doing this, let us mention some things about toric geometry. The latter is a powerful tool for studying n-dimensional complex manifolds exhibiting toric circle actions $U(1)^n$ which allow us to encode the geometric properties of the complex spaces in terms of simple combinatorial data of polytopes Δ of the R^n real space [22–25]. The simple example of toric varieties is the complex plane $\mathbf{C}$. The latter admits a $U(1)$ toric action

$$z \to z\,\mathrm{e}^{\mathrm{i}\theta} \tag{4.1}$$

which has a fixed point at $z = 0$. Thus the toric geometry of $\mathbf{C}$ can be viewed as a circle fibred on a half line parameterized by $|z|$. The circle, which is determined by the action of θ, shrinks at $z = 0$. This realization can be generalized easily to $\mathbf{C}^n$ space where we have a T^n fibration, parameterized by the angular coordinates θ_i, over an n-dimensional real base parameterized by $\left|z_i^2\right|$. The second example we want to give is the $\mathbf{CP^1}$ projective space. This space also has a $U(1)$ toric action having two fixed points describing respectively north and south poles of the two sphere $S^2 \sim \mathbf{CP^1}$. Thus the toric geometry of $\mathbf{CP^1}$ is given by an interval fibred by S^1 with zero size at the endpoints of the interval. Using these ideas, the cotangent bundle over $\mathbf{CP^1}$ can also be viewed as a toric space. In this way, we have two circle actions on this space. The first one corresponds to the action on the $\mathbf{CP^1}$ base space and the other circle acts on the fibre cotangent direction. Our next example will be the two complex dimensional projective space $\mathbf{CP^2}$. The latter has a $U(1)^2$ toric action exhibiting three fixed points defining a triangle in the $\mathbf{R^2}$ real space. The toric geometry of this manifold is described by a triangle of $\mathbf{R^2}$ fibred by a two real-dimensional torus T^2 which degenerates to an S^1 circle on the three edges and shrinks to a point on the endpoints. The cotangent bundle over $\mathbf{CP^2}$ is a 4D (eight real) local toric geometry, where we have two extra circle actions coming from the fibre cotangent directions. Note that this analysis is similar to the $\mathbf{WCP^2}$, in particular $\mathbf{WCP^2_{(1,2,1)}}$, and can be extended easily to higher dimensional (weighted) projective spaces. In what follows, we will consider the above toric geometry circle actions to identify the $U(1)$ group symmetry of quotient spaces $X_7 = \frac{X_8}{U(1)}$.

Let us consider the simple example of the $U(1)$ gauge theory with three hypermultiplets. In this case, the geometry X_8 can be viewed as $\mathbf{C^2}$ bundle over a $\mathbf{WCP^2_{(1,2,1)}}$. This manifold has four toric geometry circle actions $U(1)^2_{\text{base}} \times U(1)^2_{\text{fibre}}$; two of them correspond to the $\mathbf{WCP^2_{(1,2,1)}}$ base space denoted by $U(1)^2_{\text{base}}$ while the remaining ones $U(1)^2_{\text{fibre}}$ act on the fibre cotangent directions. In what follows, we want to divide by one finite circle toric geometry action for obtaining seven real manifolds. Mimicking the analysis of [11] and identifying the $U(1)$ group symmetry of the quotient with one finite fibre circle action

$$X_7 = \frac{X_8}{U(1)_{\text{fibre}}} \tag{4.2}$$

we can obtain a seven-dimensional geometry. Since $\frac{\mathbf{C^2}}{U(1)} = \mathbf{R}^+ \times \mathbf{C}$, the quotient space is now an $\mathbf{R}^+ \times \mathbf{C}$ bundle over a $\mathbf{WCP^2_{(1,2,1)}}$. By compactifying the $\mathbf{C}$ complex plane, which can be done by adding a point at infinity, this space will be an $\mathbf{R}^+ \times S^2$ bundle over $\mathbf{WCP^2_{(1,2,1)}}$. As in [11], this geometry is a cone on an S^2 bundle over $\mathbf{WCP^2_{(1,2,1)}}$ of G_2 holonomy. More generally, if we consider the $U(1)^r$ gauge theory with the A_r Cartan matrix gauge charges and $(r+2)$ hypermultiplets, then the quotient space is a cone on an S^2 bundle over r intersecting $\mathbf{WCP^2_{(1,2,1)}}$ weighted projective spaces according to the A_r Dynkin diagram.

Finally, a naive way to study the singularities of these X_7 manifolds is to consider the identification structure of the weighted projective spaces. The latter are not generally smooth because non-trivial fixed points under the variable identifications lead to singularities. To see this feature, consider the identification structure of $\mathbf{WCP}^{\mathbf{2}}_{(\mathbf{1},\mathbf{2},\mathbf{1})}$ defined by introducing three homogeneous complex coordinates z_1, z_2, z_3 not all of them simultaneously zero with a projective relation:

$$(z_1, z_2, z_3) \equiv \left(\lambda z_1, \lambda^2 z_2, \lambda z_3\right). \tag{4.3}$$

Note, in passing, that these (z_1, z_2, z_3) homogeneous complex coordinates can be related respectively to ψ_1, ψ_2 and ψ_3 fields of the sigma model construction. Finally, it is not hard to show that this space is singular. Indeed, if we take $\lambda = -1$, equation (4.3) reduces to

$$(z_1, z_2, z_3) \equiv (-z_1, z_2, -z_3) \tag{4.4}$$

and so we have a Z_2 orbifold singularity at $(z_1, z_2, z_3) = (0, z_2, 0)$.

5. Conclusion

In this paper, we have contributed in the M-theory compactifications to four dimensions. This involves the compactification on seven manifolds of G_2 holonomy group, leading to 4D $N = 1$ supersymmetric models. In particular, we have constructed a new class of toric hyper-Kahler eight manifolds giving G_2 holonomy spaces after dividing by one finite toric geometry circle action. This has been proceeded in two steps. We have first introduced the ADE Cartan matrices as matrix gauge charges in the 2D $N = 4$ field theoretical construction of toric hyper-Kahler eight manifolds X_8. In particular, the solution for the A_r Lie algebra is described by the cotangent bundle over r intersecting $\mathbf{WCP}^{\mathbf{2}}_{(\mathbf{1},\mathbf{2},\mathbf{1})}$ weighted projective spaces according to the A_r Dynkin diagram. Actually these spaces may extend the geometry of A_r ALE space, described by the 2D $N = 2$ type IIA superstring sigma model used in the geometric engineering method. Second, we have used the toric geometry circle actions of X_8 to build quotients $X_7 = \frac{X_8}{U(1)}$ of G_2 holonomy group.

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References

[1] Witten E 1995 String theory dynamics in various dimensions *Nucl. Phys.* B **443** 85 (*Preprint* hep-th/9503124)
[2] Townsend P K 1995 The eleven-dimensional supermembrane revisited *Phys. Lett.* B **350** 184 (*Preprint* hep-th/9501068)
[3] Gomez C 2001 *Workshop on Noncommutative Geometry, Superstrings and Particle Physics (Rabat, Morocco, May 11–12)*
[4] Townsend P K and Papadopoulos G 1995 Compactification of D = 11 supergravity on spaces of exceptional holonomy *Phys. Lett.* B **357** 472 (*Preprint* hep-th/9506150)
[5] Acharya B 2000 On realizing $N = 1$ super Yang–Mills in M-theory *Preprint* hep-th/0011089
[6] Atiyah M F and Witten E 2001 M-theory dynamics on a manifold of G_2 Holonomy *Preprint* hep-th/0107177
[7] Cvetic M, Shiu G and Uranga A M 2001 Chiral type II orientifold constructions as M-theory on G_2 holonomy spaces *Preprint* hep-th/0111179
[8] Cvetic M, Gibbons G W, Lu H and Pope C N 2001 A G_2 Unification of the deformed and resolved conifolds *Preprint* hep-th/0112138

[9] Cvetic M, Gibbons G W, Lu H and Pope C N 2001 M-theory conifolds *Preprint* hep-th/0112098
[10] Witten E 2001 Anomaly cancellation on manifolds of G_2 holonomy *Preprint* hep-th/0108165
[11] Acharya B and Witten E 2001 Chiral fermions from manifolds of G_2 holonomy *Preprint* hep-th/0109152
[12] Katz S and Vafa C 1997 *Nucl. Phys.* B **497** 146–54 (*Preprint* hep-th/9606086)
[13] Katz S, Mayr P and Vafa C 1998 Mirror symmetry and exact solution of 4d $N = 2$ gauge theories I *Adv. Theor. Math. Phys.* **1** 53
[14] Mayr P 1999 Geometric construction of $N = 2$ gauge theories *Spring Workshop on Superstrings and Related Matters (ICTP, Trieste, March 1999)*
[15] Belhaj A, Fallah A EL and Saidi E H 1999 On non simply laced mirror geometries in type II strings *Class. Quantum Grav.* **17** 515–32
[16] Gukov S, Vafa C and Witten E 2000 CFT's from Calabi–Yau four-folds *Nucl. Phys.* B **584** 69–108 (*Preprint* hep-th/9906070); Erratum 2001 B **608** 477–78
[17] Hitchin N J, Karlhede A, Lindström U and Roček M 1987 *Commun. Math. Phys.* **108** 535–89
[18] Belhaj A and Saidi E H 2001 Hyper-Kahler singularities in superstrings compactification and 2d $N = 4$ conformal field theory *Class. Quantum Grav.* **18** 57–82 (*Preprint* hep-th/0002205)
Belhaj A and Saidi E H 2000 On hyperKahler singularities *Mod. Phys. Lett.* A **15** 1767–79 (*Preprint* hep-th/0007143)
[19] Grenne B R 1996 String theory on Calabi–Yau manifolds *TASI Lectures 1996* (*Preprint* hep-th/9702155)
[20] Theisen S 2001 Introduction to Calabi–Yau manifolds *Spring School on Superstring Theories and Related Matters (ICTP, Trieste, Italy)*
[21] Morrison D R and Vafa C 1996 Compactifications of F-theory on Calabi–Yau threefolds II *Nucl. Phys.* B **476** 437–69 (*Preprint* hep-th/9603161)
[22] Vafa C 1998 On $N = 1$ Yang–Mills in four dimensions *Adv. Theor. Math. Phys.* **2** 497–503
[23] Fulton W 1993 Introduction to toric varieties *Annals of Mathematical Studies* vol 131 (Princeton, NJ: Princeton University Press)
[24] Leung N C and Vafa C 1998 *Adv. Theor. Math. Phys.* **2** 91 (*Preprint* hep-th/9711013)
[25] Belhaj A and Saidi E H 2000 Toric geometry, enhanced nonsimply laced gauge symmetries in superstrings and F-theory compactifications *Preprint* hep-th/0012131
[26] Belhaj A 2002 Mirror symmetry and Landau–Ginzburg Calabi–Yau superpotentials in F-theory compactifications *J. Phys. A: Math. Gen.* **35** 965–84 (*Preprint* hep-th/0112005)